

MATHEMATICS



Application of integrals

1. **Elementary area:** The area is called elementary area which is located at any arbitrary position within the region which is specified by some value of x between a and b .
2. The area of the region bounded by the curve $y = f(x)$, x -axis and the lines $x = a$ and $x = b$ ($b > a$) is given by the formula: $\text{Area} = \int_a^b y dx = \int_a^b f(x) dx$
3. The area of the region bounded by the curve $x = \theta(y)$, y -axis and the lines $y = c$, $y = d$ is given by the formula: $\text{Area} = \int_c^d x dy = \int_c^d \theta(y) dy$
4. The area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by the formula, $\text{Area} = \int_a^b [f(x) - g(x)] dx$, where $f(x) \geq g(x)$ in $[a, b]$.
5. If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then we write the areas as:

$$\text{Area} = \int_a^b [f(x) - g(x)] dx + \int_c^b [g(x) - f(x)] dx$$

MIND MAP : LEARNING MADE SIMPLE

CHAPTER - 8

Application of the Integrals

Area between two curves

The area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x=a$, $x=b$ is given by

$$A = \int_a^b [f(x) - g(x)] dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b]$$

For eg: To find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$ $\Rightarrow y = \sqrt{x} = f(x)$, and $y = x^2 = g(x)$ $\Rightarrow y^2 = x$ and $y^2 = x^2$ $\Rightarrow y = x$, and $y = x^2 = g(x)$, where $f(x) \geq g(x)$ in $[0, 1]$.

$$\begin{aligned} \text{Area, } A &= \int_0^1 [f(x) - g(x)] dx \\ &= \int_0^1 [\sqrt{x} - x^2] dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3} \right]_0^1 \\ &= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ Sq. units.} \end{aligned}$$

The area of the region bounded by the curve $y = f(x)$, x -axis and the lines $x=a$ and $x=b$ ($b > a$) is given by

$$A = \int_a^b y dx \text{ or } \int_a^b f(x) dx.$$

For eg: the area bounded by $y = x^2$, x -axis in I quadrant and the lines $x=2$ and $x=3$ is -

$$A = \int_2^3 y dx = \int_2^3 x^2 dx = \left[\frac{x^3}{3} \right]_2^3 = \frac{1}{3}(27 - 8) = \frac{19}{3} \text{ Sq. units.}$$

The area of the region bounded by the curve $x = f(y)$, y -axis and the lines $y=c$ and $y=d$ ($d > c$) is given by $A = \int_c^d x dy$ or $\int_c^d f(y) dy$.

For eg: the area bounded by $x = y^3$, y -axis in the I quadrant and the lines $y=1$ and $y=2$ is

$$A = \int_1^2 x dy = \int_1^2 y^3 dy = \left[\frac{1}{4}y^4 \right]_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4} \text{ Sq. units}$$

Important Questions

Multiple Choice questions-

1. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(a) π

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{4}$

2. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(a) 2

(b) $\frac{9}{4}$

(c) $\frac{9}{3}$

(d) $\frac{9}{2}$

3. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(a) $2(\pi - 2)$

(b) $\pi - 2$

(c) $2\pi - 1$

(d) $2(\pi + 2)$.

4. Area lying between the curves $y^2 = 4x$ and $y = 2$ is:

(a) $\frac{2}{3}$

(b) $\frac{1}{3}$

(c) $\frac{1}{4}$

(d) $\frac{3}{4}$

5. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(a) -9

(b) $-\frac{15}{4}$

(c) $\frac{15}{4}$

(d) $\frac{17}{4}$

6. The area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

(a) 0

(b) $-\frac{1}{3}$

(c) $\frac{2}{3}$

(d) $\frac{4}{3}$

7. The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

(a) $\frac{4}{3}(4\pi - \sqrt{3})$

(b) $\frac{1}{3}(4\pi + \sqrt{3})$

(c) $\frac{2}{3}(8\pi - \sqrt{3})$

(d) $\frac{4}{3}(8\pi + \sqrt{3})$

8. The area enclosed by the circle $x^2 + y^2 = 2$ is equal to

(a) 4π sq. units

(b) $2\sqrt{2}\pi$ sq. units

(c) $4\pi^2$ sq. units

(d) 2π sq. units.

9. The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to

(a) $\pi^2 ab$

(b) πab (c) $\pi a^2 b$ (d) πab^2 .

10. The area of the region bounded by the curve $y = x^2$ and the line $y = 16$ is

(a) $\frac{32}{3}$ (b) $\frac{256}{3}$ (c) $\frac{64}{3}$ (d) $\frac{128}{3}$

Very Short Questions:

1. Find the area of region bounded by the curve $y = x^2$ and the line $y = 4$.

2. Find the area bounded by the curve $y = x^3$, $x = 0$ and the ordinates $x = -2$ and $x = 1$.

3. Find the area bounded between parabolas $y^2 = 4x$ and $x^2 = 4y$.

4. Find the area enclosed between the curve $y = \cos x$, $0 \leq x \leq \frac{\pi}{4}$ and the co-ordinate axes.

5. Find the area between the x-axis curve $y = \cos x$ when $0 \leq x < 2$.

6. Find the ratio of the areas between the center $y = \cos x$ and $y = \cos 2x$ and x-axis for $x = 0$ to

$$x = \frac{\pi}{3}$$

7. Find the areas of the region:

$$\{(x, y) : x^2 + y^2 \leq 1 \leq x + 4\}$$

Long Questions:

1. Find the area enclosed by the circle:

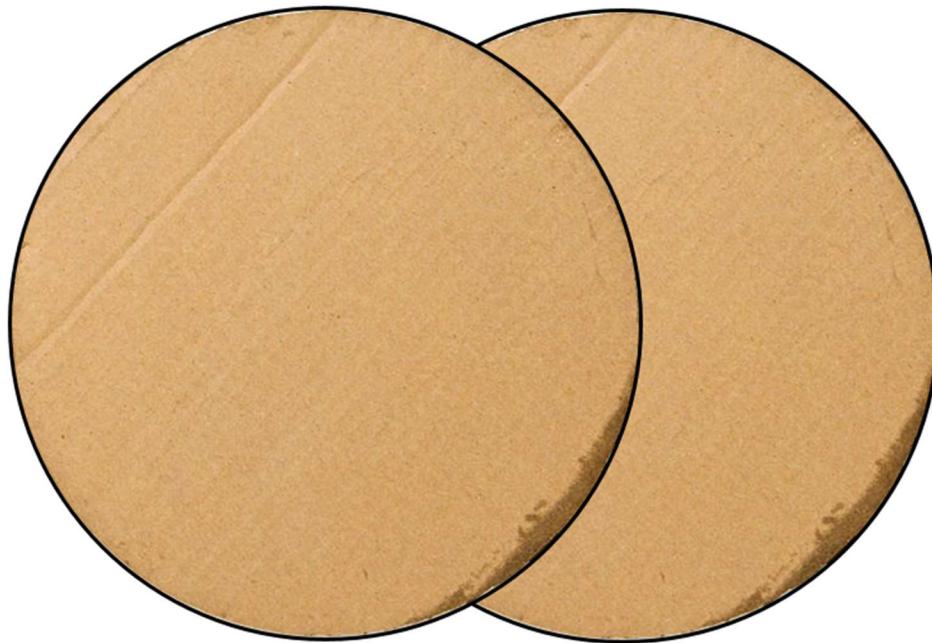
$$x^2 + y^2 = a^2. \text{ (N.C.E.R.T.)}$$

2. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. (C.B.S.E. 2018)

3. Find the area bounded by the curves $y = \sqrt{x}$, $2y + 3 = Y$ and Y-axis. (C.B.S.E. Sample Paper 2018-19)
4. Find the area of region:
 $\{(x,y): x^2 + y^2 < 8, x^2 < 2y\}$. (C.B.S.E. Sample Paper 2018-19)

Case Study Questions:

1. Ajay cut two circular pieces of cardboard and placed one upon other as shown in figure. One of the circle represents the equation $(x - 1)^2 + y^2 = 1$, while other circle represents the equation $x^2 + y^2 = 1$.



Based on the above information, answer the following questions.

- i. Both the circular pieces of cardboard meet each other at

a. $x = 1$

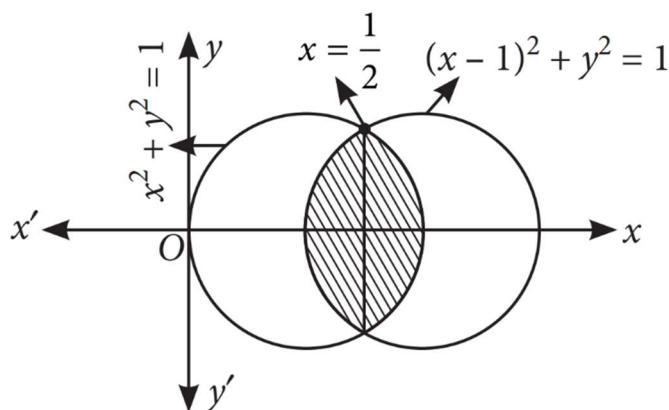
b. $x = \frac{1}{2}$

c. $x = \frac{1}{3}$

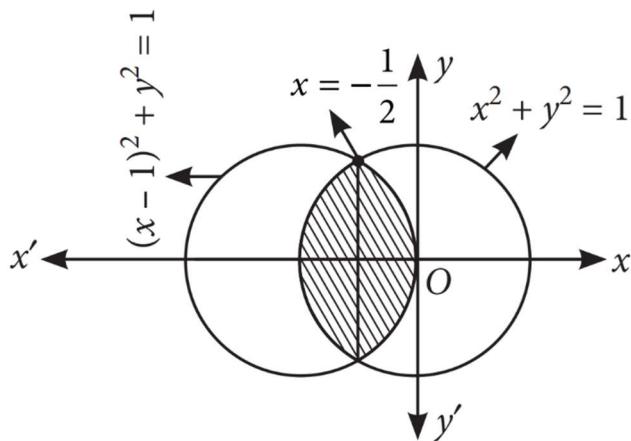
d. $x = \frac{1}{4}$

- ii. Graph of given two curves can be drawn as.

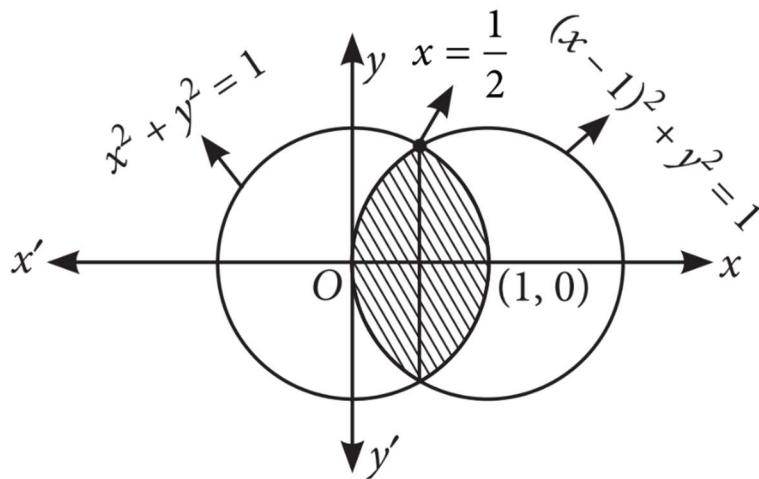
a.



b.



c.



d. None of these

iii. Value of $\int_0^{\frac{1}{2}} \sqrt{1 - (x - 1)^2} dx$ is.

a. $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$

b. $\frac{\pi}{6} + \frac{\sqrt{3}}{8}$

c. $\frac{\pi}{2} + \frac{\sqrt{3}}{4}$

d. $\frac{\pi}{2} - \frac{\sqrt{3}}{4}$

iv. Value of $\int_{\frac{1}{2}}^1 \sqrt{1 - x^2} dx$ is.

a. $\frac{\pi}{6} + \frac{\sqrt{3}}{4}$

b. $\frac{\pi}{6} + \frac{\sqrt{3}}{8}$

c. $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$

d. $\frac{\pi}{2} - \frac{\sqrt{3}}{4}$

v. Area of hidden portion of lower circle is.

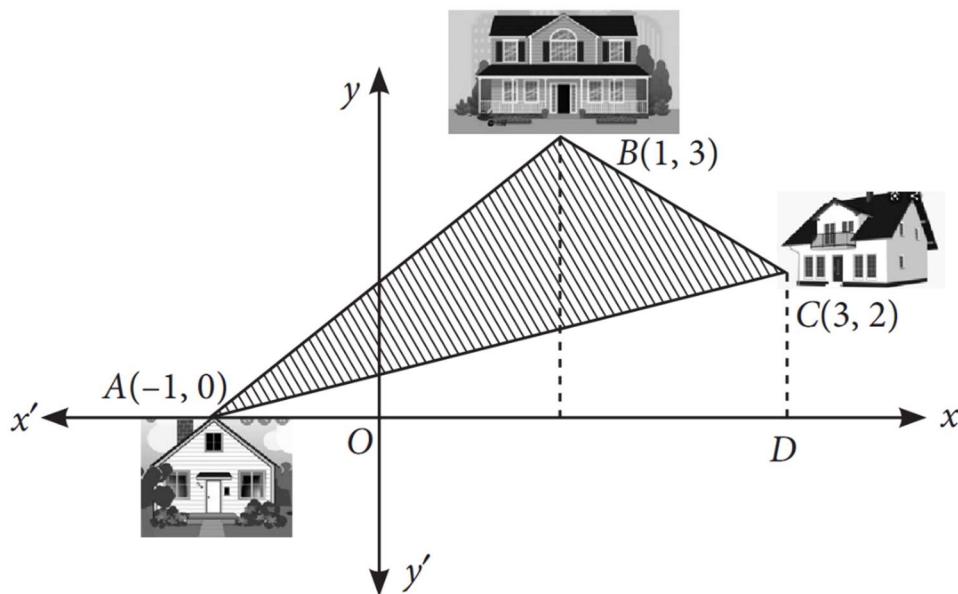
a. $\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} \right)$ sq.units

b. $\left(\frac{\pi}{3} - \frac{\sqrt{3}}{8} \right)$ sq.units

c. $\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8} \right)$ sq.units

d. $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right)$ sq.units

2. Location of three houses of a society is represented by the points $A(-1, 0)$, $B(1, 3)$ and $C(3, 2)$ as shown in figure.



Based on the above information, answer the following questions

(i) Equation of line AB is.

a. $y = \frac{3}{2}(x + 1)$

b. $y = \frac{3}{2}(x - 1)$

c. $y = \frac{1}{2}(x + 1)$

d. $y = \frac{1}{2}(x - 1)$

(ii) Equation of line BC is.

a. $y = \frac{1}{2}x - \frac{7}{2}$

b. $y = \frac{3}{2}x - \frac{7}{2}$

c. $y = \frac{-1}{2}x + \frac{7}{2}$

d. $y = \frac{3}{2}x + \frac{7}{2}$

(iii) Area of region ABCD is.

- a. 2 sq. units
- b. 4 sq. units
- c. 6 sq. units
- d. 8 sq. units

(iv) Area of $\triangle ADC$ is,

- a. 4 sq. units
- b. 8 sq. units
- c. 16 sq. units
- d. 32 sq. units

(v) Area of $\triangle ABC$ is.

- a. 3 sq. units
- b. 4 sq. units
- c. 5 sq. units
- d. 6 sq. units

Answer Key-

Multiple Choice questions-

1. Answer: (a) π
2. Answer: (a) 2
3. Answer: (b) $\pi - 2$
4. Answer: (b) $\frac{1}{3}$
5. Answer: (b) $-\frac{15}{4}$
6. Answer: (c) $\frac{2}{3}$
7. Answer: (c) $\frac{2}{3} (8\pi - \sqrt{3})$
8. Answer: (d) 2π sq. units.
9. Answer: (b) πab
10. Answer: (b) $\frac{256}{3}$

Very Short Answer:

1. Solution: $\frac{32}{2}$ sq. units.
2. Solution: $\frac{17}{4}$ sq. units.
3. Solution: $\frac{16}{3}$ sq. units.
4. Solution: $\frac{1}{2}$ sq. units.
5. Solution: 4 sq. units
6. Solution: 2 : 1.
7. Solution: $\frac{1}{2}(\pi - 1)$ sq. units.

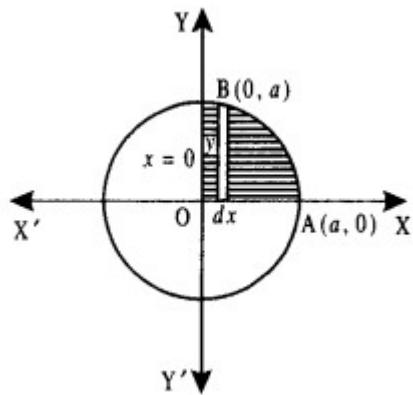
Long Answer:

1. Solution:

The given circle is

$$x^2 + y^2 = a^2 \dots\dots\dots (1)$$

This is a circle whose center is (0,0) and radius 'a'.



Area of the circle = 4 x (area of the region OABO, bounded by the curve, x-axis and ordinates $x = 0, x = a$)

[\because Circle is symmetrical about both the axes]

$$= 4 \int_0^a y dx \text{ [Taking vertical strips] } 0$$

$$= 4 \int_0^a \sqrt{a^2 - x^2} dx$$

$$[\because (1) \Rightarrow y = \pm \sqrt{a^2 - x^2}]$$

But region OABO lies in 1st quadrant, $\therefore y$ is +ve]

$$\begin{aligned} &= 4 \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\ &= 4 \left[\left\{ \frac{a}{2} (0) + \frac{a^2}{2} \sin^{-1} (1) \right\} - \{0 + 0\} \right] \\ &= 4 \left(\frac{a^2}{2} \cdot \frac{\pi}{2} \right) = \pi a^2 \text{ sq. units.} \end{aligned}$$

2. Solution:

We have:

$$y = x \dots (1)$$

$$\text{and } x^2 + y^2 = 32 \dots (2)$$

(1) is a st. line, passing through (0,0) and (2) is a circle with centre (0,0) and radius $4\sqrt{2}$ units.
Solving (1) and (2) :

Putting the value of y from (1) in (2), we get:

$$x^2 + x^2 = 32$$

$$2x^2 = 32$$

$$x^2 = 16$$

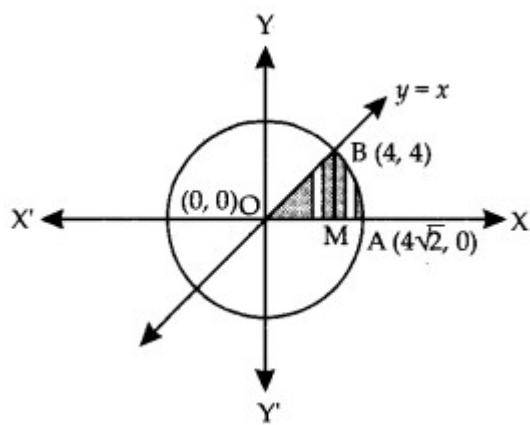
$$x = 4.$$

$[\because$ region lies in first quadrant]

Also, $y = 4$

Thus, the line (1) and the circle (2) meet each other at B (4,4), in the first quadrant.

Draw BM perp. to x – axis.



∴ Reqd. area = area of the region OMBO + area of the region BMAB ... (3)

Now, area of the region OMBO

$$\begin{aligned}
 &= \int_0^4 y \, dx \quad [\text{Taking vertical strips}] \\
 &= \int_0^4 x \, dx = \left[\frac{x^2}{2} \right]_0^4 = \frac{1}{2} (16 - 0) = 8.
 \end{aligned}$$

Again, area of the region BMAB

$$\begin{aligned}
 &= \int_4^{4\sqrt{2}} y \, dx \quad [\text{Taking vertical strips}] \\
 &= \int_4^{4\sqrt{2}} \sqrt{32 - x^2} \, dx \\
 &[\because y^2 = 32 - x^2 \Rightarrow y = \sqrt{32 - x^2}, \text{ taking +ve sign, as it lies in 1st quadrant}] \\
 &= \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2} \, dx \\
 &= \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2} \sin^{-1} \frac{x}{4\sqrt{2}} \right]_4^{4\sqrt{2}} \\
 &= \left\{ \frac{1}{2} 4\sqrt{2} \times 0 + \frac{32}{2} \sin^{-1} (1) \right\} \\
 &\quad - \left\{ \frac{4}{2} \sqrt{32 - 16} + \frac{32}{2} \sin^{-1} \frac{1}{\sqrt{2}} \right\} \\
 &= 0 + 16 \left(\frac{\pi}{2} \right) - \left(2 \times 4 + 16 \times \frac{\pi}{4} \right)
 \end{aligned}$$

$$= 8\pi - (8 + 4\pi) = 4\pi - 8$$

∴ From (3),

$$\text{Required area} = 8 + (4\pi - 8) = 4\pi \text{ sq. units.}$$

3. Solution:

The given curves are

$$y = \sqrt{x} \dots \dots \dots (1)$$

$$\text{and } 2y + 3 = x \dots(2)$$

Solving (1) and (2), we get;

$$\sqrt{2y+3} = y$$

Squaring, $2y + 3 = y^2$

$$\Rightarrow y^2 - 2y - 3 = 0$$

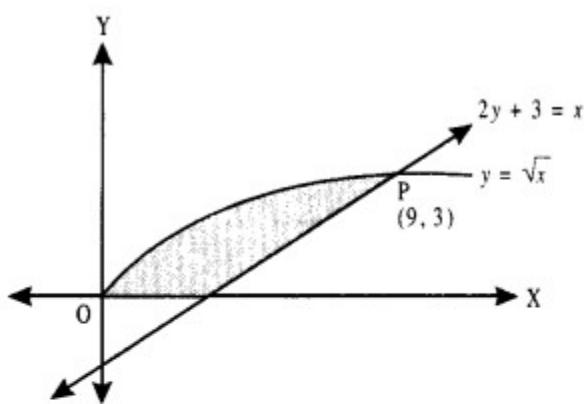
$$\Rightarrow (y + 1)(y - 3) = 0 \Rightarrow y = -1, 3$$

$$\Rightarrow y = 3 \quad [\because y > 0]$$

Putting in (2),

$$x = 2(3) + 3 = 9.$$

Thus, (1) and (2) intersects at (9, 3).



$$\therefore \text{Reqd. Area} = \int_0^3 (2y + 3) dy - \int_0^3 y^2 dy$$

$$= [y^2 + 3y]_0^3 - \left[\frac{y^3}{3} \right]_0^3$$

$$= (9 + 9) - \left(\frac{27}{3} \right)$$

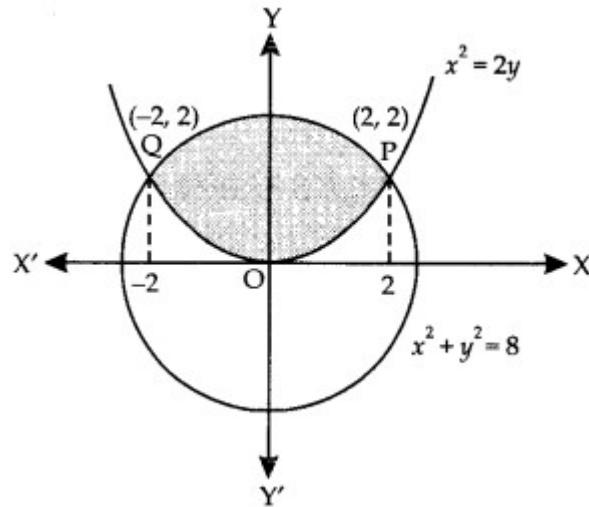
$$= 9 + 9 - 9 = 9 \text{ sq. units.}$$

4. Solution:

The given curves are;

$$x^2 + y^2 = 8 \dots\dots\dots (1)$$

$$x^2 = 2y \dots\dots\dots (2)$$



Solving (1) and (2):

$$8 - y^2 = 2y$$

$$\Rightarrow y^2 + 2y - 8 = 0$$

$$\Rightarrow (y + 4)(y - 2) = 0$$

$$= y = -4, 2$$

$$\Rightarrow y = 2. [\because y > 0]$$

$$\text{Putting in (2), } x^2 = 4$$

$$\Rightarrow x = -2 \text{ or } 2.$$

Thus, (1) and (2) intersect at P(2, 2) and Q(-2, 2).

$$\therefore \text{Required area} = \int_{-2}^2 \sqrt{8 - x^2} dx - \int_{-2}^2 \frac{x^2}{2} dx$$

$$= 2 \left[\int_0^2 \sqrt{(2\sqrt{2})^2 - x^2} dx - \int_0^2 \frac{x^2}{2} dx \right]$$

$$= 2 \left[\frac{x\sqrt{8-x^2}}{2} + \frac{8}{2} \sin^{-1} \left(\frac{x}{2\sqrt{2}} \right) \right]_0^2 - \frac{1}{3} [x^3]_0^2$$

$$= 2 \left[2 + 4 \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) - 0 \right] - \frac{1}{3} [8 - 0]$$

$$= 4 + 8 \left(\frac{\pi}{4} \right) - \frac{8}{3} = \left(2\pi + \frac{4}{3} \right) \text{sq. units.}$$

Case Study Answers:

1. Answer :

$$\text{i. (b)} \quad x = \frac{1}{2}$$

Solution:

$$\text{We have, } (x - 1)^2 + y^2 = 1$$

$$\Rightarrow y = \sqrt{1 - (x - 1)^2}$$

$$\text{Also } x^2 + y^2 = 1$$

$$\Rightarrow y = \sqrt{1 - x^2}$$

From (i) and (ii), we get

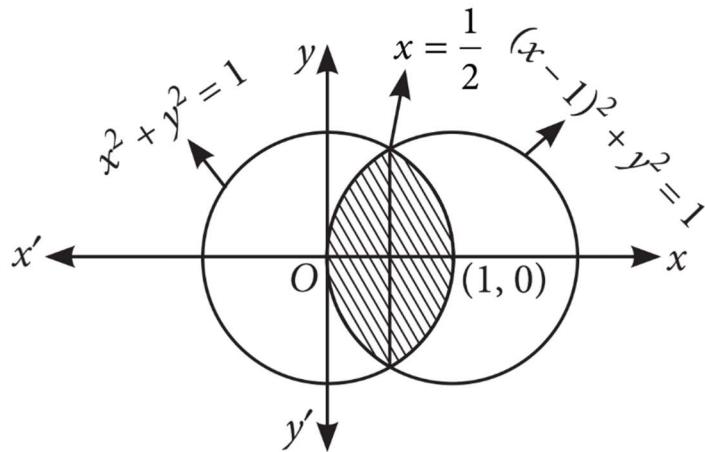
$$\sqrt{1 - (x - 1)^2} = \sqrt{1 - x^2}$$

$$\Rightarrow (x - 1)^2 = x^2$$

$$\Rightarrow 2x = 1$$

$$\Rightarrow x = \frac{1}{2}$$

ii. (c)



$$\text{iii. (a)} \quad \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

Solution:

$$\begin{aligned}
 & \left[\int_0^{\frac{1}{2}} \sqrt{1 - (x-1)^2} + \frac{1}{2} \sin^{-1} \left(\frac{x-1}{1} \right) \right] \\
 &= \frac{1}{2} \left(\frac{1}{2} - 1 \right) \sqrt{1 - \frac{1}{4}} + \frac{1}{2} + \sin^{-1} \left(-\frac{1}{2} \right) - \left(-\frac{1}{2} \right) \\
 &\quad - \frac{1}{2} \sin^{-1} \\
 &= \left[\frac{-1}{4} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\pi}{6} + 0 + \frac{1}{2} \cdot \frac{\pi}{2} \right] = \frac{\sqrt{3}}{8} - \frac{\pi}{12} + \frac{\pi}{4} \\
 &= \frac{\pi}{6} - \frac{\sqrt{3}}{8}
 \end{aligned}$$

$$\text{iv. (c)} \quad \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

$$\begin{aligned}
 \int_{\frac{1}{2}}^1 \sqrt{1 - x^2} dx &= \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x \right]_{\frac{1}{2}}^1 \\
 &= 0 + \frac{1}{2} \sin^{-1}(1) - \frac{1}{4} \sqrt{1 - \frac{1}{4}} - \frac{1}{2} \sin^{-1} \left(\frac{1}{2} \right) \\
 &= \frac{\pi}{4} - \frac{\sqrt{3}}{8} - \frac{\pi}{12} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}
 \end{aligned}$$

$$\text{v. (d)} \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \text{ sq.units}$$

Solution:

$$= 2 \left[\int_0^{\frac{1}{2}} \sqrt{1 - (x-1)^2} dx + \int_{\frac{1}{2}}^1 \sqrt{1 - x^2} dx \right]$$

$$= 2 \left[\frac{\pi}{6} - \frac{\sqrt{3}}{8} + \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right]$$

$$= 2 \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \text{ sq.units}$$

2. Answer :

$$\text{i. (a)} \quad y = \frac{3}{2}(x+1)$$

Solution:

$$\text{Equation of line AB is. } y - 0 = \frac{3-0}{1+1}(x+1) \Rightarrow y = \frac{3}{2}(x+1)$$

$$\text{ii. (c)} \quad y = \frac{-1}{2}x + \frac{7}{2}$$

Solution:

$$\text{Equation of line BC is } y - 3 = \frac{2-3}{3-1}(x+1)$$

$$\Rightarrow y = -\frac{1}{2}x + \frac{1}{2} + 3 \Rightarrow y = \frac{-1}{2}x + \frac{7}{2}$$

iii. (d) 8 sq. units

Solution:

$$\text{Area of region } ABCD = \text{Area of } \triangle ABE + \text{Area of region BCDE}$$

$$\begin{aligned} &= \int_{-1}^1 \frac{3}{2}(x+1)dx + \int_1^3 \left(\frac{-1}{2}x + \frac{7}{2} \right) dx \\ &= \frac{3}{2} \left[\frac{x^2}{2} + x \right]_{-1}^1 + \left[\frac{-x^2}{4} + \frac{7}{2}x \right]_1^3 \\ &= \frac{3}{2} \left[\frac{1}{2} + 1 - \frac{1}{2} + 1 \right] + \left[\frac{-9}{4} + \frac{21}{2} + \frac{1}{4} - \frac{7}{2} \right] \\ &= 3 + 5 = 8 \text{ sq. units} \end{aligned}$$

iv. (a) 4 sq. units

Solution:

$$\text{Equation of line } AC \text{ is } y - 0 = \frac{2-0}{3+1}(x+1)$$

$$\Rightarrow y = \frac{1}{2}(x+1)$$

$$\begin{aligned} \therefore \text{Area of } \triangle ADC &= \int_{-1}^3 \frac{1}{2}(x+1)dx = \left[\frac{x^2}{4} + \frac{1}{2}x \right]_{-1}^3 \\ &= \frac{9}{4} + \frac{3}{2} - \frac{1}{4} + \frac{1}{2} = 4 \text{ sq. units} \end{aligned}$$

v. (b) 4 sq. units

Solution:

$$\text{Area of } \triangle ABC = \text{Area of region } ABCD - \text{Area of } \triangle ACD = 8 - 4 = 4 \text{ sq. units}$$